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THEORETICAL INVESTIGATION OF THE EFFECT OF VARIATION
OF ATMOSPHERIC DENSITY WITH TIME ON THE
STABILITY OF AN HYPOTHETICAL
AIRCRAFT HAVING A SINGLE
DEGREE OF FREEDOM

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1. Variation of Angular Displacement With Time for a Single-Degree-of-Freedom System With Zero Aerodynamic Damping in an Atmosphere the Density of Which Is Increasing Exponentially With Time 13

LIST OF SYMBOLS

a	constant in expression $\rho = \rho_{SL} e^{-ah_o - b\Delta h}$
b	constant in expression $\rho = \rho_{SL} e^{-ah_o - b\Delta h}$
c	constant in expression $\rho = \rho_{SL} e^{-ah_o \pm ct}$
e	base of natural logarithms
f	unknown function of time
g	unknown function of time
h_o	height above sea level at time $t = 0$, feet
Δh	height above h_o, feet
k	designation of general term in hypergeometric series
l	reference length on aircraft, feet
t	time, seconds
A	constant proportional to aerodynamic damping

$$A = \frac{\frac{1}{2}V^2 S l \frac{l}{2V} \rho_{SL} e^{-ah_o}\left(-C_{m_{\dot{\theta}}}\right)}{I}$$

B	constant proportional to aerodynamic spring rate

$$B = \frac{\frac{1}{2}V^2 S l \rho_{SL} e^{-ah_o}\left(-C_{m_{\theta}}\right)}{I}$$

C_1, C_2	constants of integration

C_m aerodynamic moment coefficient

$$C_m = \frac{M}{\frac{1}{2}\rho V^2 S l}$$

C_{m_θ} partial derivative of aerodynamic moment coefficient with respect to θ

$$C_{m_\theta} = \frac{\partial C_m}{\partial \theta}$$

$C_{m_{\dot\theta}}$ partial derivative of aerodynamic moment coefficient with respect to $\left(\frac{\dot\theta l}{2V}\right)$

$$C_{m_{\dot\theta}} = \frac{\partial C_m}{\partial\left(\frac{\dot\theta l}{2V}\right)}$$

$F\left(\frac{A}{Bc}, 1, \xi_I\right)$ hypergeometric series

$F^*\left(\frac{A}{Bc}, 1, \xi_I\right)$ modified hypergeometric series

I moment of inertia of aircraft about axis of rotation, slugs ft^2

I_o modified Bessel function of first kind of order zero

J_n Bessel function of order n

K_o modified Bessel function of second kind of order zero

$K_1, K_2, \ldots, K_8$ arbitrary constants

M	aerodynamic moment about axis of rotation
M_θ	partial derivative of aerodynamic moment with respect to θ
$M_{\dot{\theta}}$	partial derivative of aerodynamic moment with respect to $\dot{\theta}$
S	reference area
V	flight velocity, feet per second
V_h	vertical component of flight velocity, feet per second
η	dependent variable such that $\eta(\xi) = \theta(t)$
η_1	dependent variable such that $\eta_1(\xi_I) = \eta(\xi)$
θ	angular displacement from reference attitude
θ_o	value of θ when $t = 0$
λ	constant in expression $\theta = ke^{\lambda t}$
ξ	independent variable $\xi = e^{\pm ct}$
ξ_1	independent variable $\xi_1 = -\frac{A}{c}\xi$
ρ	air density, slugs per cubic foot
ρ_{SL}	air density at sea level
σ	frequency parameter

CHAPTER I

INTRODUCTION

An aircraft or missile traveling along an inclined flight path is subjected to a variation of air density with time. The forces and moments exerted by the air on the body envelope and on any supporting, stabilizing, or control surface are proportional to the air density. Any deviation of the aircraft from the attitude corresponding to equilibrium in steady rectilinear flight is a time-dependent motion. It is apparent that if there is a substantial rate of change of air density with time, this rate of change must be taken into account in calculating the aircraft motion. It is of particular interest to determine whether the variation of density with time will adversely affect the stability of the aircraft.

This problem is becoming increasingly important with the increase in aircraft and missile speeds. Missiles and aircraft of the future may reach orbital or near orbital speeds in flight at altitudes where the atmospheric pressure is practically zero. Among the many problems which must be solved before such flight is practicable is that of insuring that the vehicle will be stable and controllable while leaving and reentering the atmosphere and hence being subjected to large rates of change of atmospheric density. It is, therefore, essential to be able to evaluate the effect upon stability of rate of change of air density with time.

That the effect of variation of density with time can not be neglected in estimating performance has been known for many years (reference 1). In 1942, Scheubel (reference 2) showed that the density variation with height may appreciably affect dynamic stability in "level" flight, i.e., flight involving small deviations from a level mean path. Neumark (reference 3) points out that to treat the general problem of the effect upon dynamic stability of the density variation with time in an inclined flight path will "require an entirely new mathematical treatment, more difficult than anything in the familiar theory of stability." The essential difference in the mathematical treatment arises from the fact that the inclusion of terms describing the effect of variation of density with time results in linear differential equations with variable coefficients, whereas the classical mathematical treatment based on the assumption of small deviations from a mean flight gives linear differential equations with constant coefficients. The solution of a set of simultaneous differential equations with variable coefficients such as is required to describe motion with more than one degree of freedom appears to present great mathematical difficulty and, so far as is known, has not been accomplished.

In the present investigation, an attempt will be made to gain some insight as to the general effect of rate of change in density with time by making a theoretical study of a simple single-degree-of-freedom system. It will be assumed that this system is subjected to an exponential variation of density with time. The resulting linear differential equation with variable coefficients for the motion following a disturbance will be set up and solved. The motion will be discussed considering both ascending and descending flight. The implications of the single-degree-of-freedom solution relative to the general case of free flight with six degrees of freedom will be considered.

The material to be presented will be arranged as follows:

(1) Development of the Differential Equation

(2) Analysis and Solution of the Differential Equation

(3) Discussion

(4) Summary

(5) Appendix I - Hypergeometric Solution for the Differential Equation

CHAPTER II

DEVELOPMENT OF THE DIFFERENTIAL EQUATION

The equation of equilibrium of inertia and aerodynamic moments for an aircraft having a single degree of freedom can be written

$$I\ddot{\theta} = M_{\dot{\theta}}\dot{\theta} + M_{\theta}\theta \qquad (1)$$

where θ is the angular deviation from the trim attitude. In writing this equation, it is assumed that the aerodynamic moments $M_{\dot{\theta}}$ and M_{θ} are linear with respect to $\dot{\theta}$ and θ, respectively, for the magnitudes to be encountered, that they are mutually independent, and that they are algebraically additive.

The partial derivative of the aerodynamic moment due to angular displacement M_{θ} can be written as

$$M_{\theta} = \frac{1}{2}\rho V^2 S l C_{m_{\theta}}$$

where S is a reference area, l is a reference length, and $C_{M_{\theta}} = \frac{\partial C_M}{\partial \theta}$ is a nondimensional coefficient dependent on the aerodynamic configuration.

Similarly, the partial derivative of the aerodynamic moment due to angular velocity $M_{\dot{\theta}}$ can be written as

$$M_{\dot{\theta}} = \frac{1}{2}\rho V^2 S l C_{m_{\dot{\theta}}} \frac{l}{2V}$$

where

$$C_{m_{\dot{\theta}}} = \frac{\partial C_m}{\partial\left(\frac{\dot{\theta} l}{2V}\right)}$$

is a nondimensional damping coefficient dependent on the aerodynamic configuration. Substituting the expanded expressions for M_θ and $M_{\dot{\theta}}$ in equation (1) gives

$$I\ddot{\theta} = \frac{1}{2}\rho V^2 S l C_{m_{\dot{\theta}}} \frac{l}{2V} \dot{\theta} + \frac{1}{2}\rho V^2 S l C_{m_\theta} \theta \qquad (2)$$

For the purposes of this paper, it will be assumed that V is a constant and that ρ is a function of time. It will be further assumed that the vehicle in question is traveling at near orbital speed on an inclined flight path, and that the change in height above the earth's surface is linear with time (the effect of the earth's curvature is neglected, or can be considered to be compensated by a corresponding curvature of the flight path in space). From Figure 3 of reference 4, it will be seen that the variation of atmospheric density with height can be approximated by the relation

$$\log_{10}\rho = -2.62 - \frac{(9.00 - 2.62)h}{400{,}000} \tag{3}$$

in the interval between sea level and $h = 400{,}000$ feet. Thus we can write

$$\rho = \rho_{SL}e^{-ah_o - b\Delta h}$$

Since it has been assumed that Δh varies linearly with time

$$\rho = \rho_{SL}e^{-ah_o}e^{\pm ct} \tag{4}$$

where $a = 3.67 \times 10^{-5}$ and $c = 3.67 \times 10^{-5}\, V_h t$ with the plus sign corresponding to descending flight and the minus sign to ascending flight. Substituting in (2)

$$I\ddot{\theta} + \frac{1}{2}\rho_{SL}V^2 S l\left(-C_{m_{\dot{\theta}}}\right)\frac{l}{2V}\,e^{-ah_o}e^{\pm ct}\dot{\theta} +$$
$$\frac{1}{2}\rho_{SL}V^2 S l\left(-C_{m_\theta}\right)e^{-ah_o}e^{\pm ct}\theta = 0$$

Letting

$$A = \frac{\frac{1}{2}V^2\rho_{SL}S l\,\frac{l}{2V}\,e^{-ah_o}\left(-C_{m_{\dot{\theta}}}\right)}{I}$$

and

$$B = \frac{\frac{1}{2}V^2\rho_{SL}S l e^{-ah_o}\left(-C_{m_\theta}\right)}{I}$$

gives the differential equation

$$\ddot{\theta} + Ae^{ct}\dot{\theta} + Be^{\pm ct}\theta = 0 \tag{5}$$

representing the motion following a disturbance of a single-degree-of-freedom system in an atmospheric density varying exponentially with time. The values of A and B obviously must be calculated using

$$\rho = \rho_{SL}e^{-ah_0}$$

corresponding to the altitude at which $t = 0$.

CHAPTER III

ANALYSIS

AND

SOLUTION OF THE DIFFERENTIAL EQUATION

The general solution of the equation

$$\ddot{\theta} + Ae^{\pm ct}\dot{\theta} + Be^{\pm ct}\theta = 0$$

is a hypergeometric series (see Appendix I) which does not afford a simple or readily understandable expression for the variation of θ with time. Since the purpose of this paper is to elucidate the motion, it appears desirable to present and study solutions which result from certain simplifying assumptions. The first assumption will be that $c \to 0$. Considering (1), when $c = 0$ gives

$$\ddot{\theta} + A\dot{\theta} + B\theta = 0 \qquad (6)$$

A solution is readily obtained by assuming the variation of θ with time to be given by

$$\theta = Ke^{\lambda t}$$

Substituting in (6) and solving for λ gives

$$\lambda = -\frac{A}{2} \pm \sqrt{\frac{A^2}{2} - B}$$

or

$$\theta = K_1 e^{\left(-\frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t} + K_2 e^{\left(-\frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t}$$

or

$$\theta = e^{-\frac{A}{2}t}\left[K_1 e^{\left(\sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t} + K_2 e^{\left(-\sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t}\right] \qquad (7)$$

If

$$\left(\frac{A}{2}\right)^2 < B$$

$$\theta = e^{-\frac{A}{2}t}\left[K_3 \cos\left(\sqrt{B - \frac{A}{2}^{2}}\right)t + K_4 \sin\left(\sqrt{B - \left(\frac{A}{2}\right)^2}\right)t\right] \qquad (7a)$$

If

$$\left(\frac{A}{2}\right)^2 = B$$

$$\theta = e^{-\frac{A}{2}t}\left[K_5 + K_6 t\right] \qquad (7b)$$

If

$$\left(\frac{A}{2}\right)^2 > B$$

$$\theta = K_7 e^{\left(-\frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t} + K_8 e^{\left(-\frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 - B}\right)t} \qquad (7c)$$

These solutions are well known and the steps have been repeated here only for completeness.

The case of greatest practical interest is that described by **equation** (7a) when A is very small, corresponding to poor aerodynamic damping. When the damping is zero, the amplitude of the oscillation remains constant and hence the energy in the oscillatory mode is constant and can be expressed as

$$E = \frac{I}{2} B \theta_m{}^2 \qquad (8)$$

where θ_m is the amplitude and BI is the aerodynamic spring rate.

$$B = \frac{\frac{1}{2} V^2 S l \left(-C_{m_\theta}\right) \rho_{SL} e^{-ah_o}}{I}$$

Since B decreases exponentially with altitude (for constant V) it is apparent that if there is no aerodynamic damping the amplitude of the oscillation must increase exponentially with altitude unless there is some energy transfer because of the change in density with time during the motion. In order to determine whether such an interchange exists, it is necessary to consider the case when $c \neq o$. It will be convenient to let $A = 0$, thus removing the energy drain due to damping.

Equation (5) with $A = 0$ has a closed analytic solution obtained as follows:

Let

$$\theta(t) = \eta(\xi)$$

where

$$\xi = e^{\pm ct}$$

then

$$\dot{\theta} = \eta'\frac{d\xi}{dt} = \pm\eta' c e^{\pm ct}$$
$$= \pm c\eta'\xi$$

$$\ddot{\theta} = c^2\xi^2\eta'' + c^2\xi\eta'$$

Substituting in (5)

$$c^2\xi^2\eta'' + c^2\xi\eta' \pm Ac\xi^2\eta' + B\xi\eta = 0$$

or

$$\xi\eta'' + \left(1 \pm \frac{A}{c}\xi\right)\eta' + \frac{B}{c^2}\eta = 0 \qquad (9)$$

Letting $A = 0$ gives

$$\xi\eta'' + \eta' + \frac{B}{c^2}\eta = 0 \tag{10}$$

Equation (10) can also be written as

$$\frac{d(\xi\eta')}{d\xi} + \frac{B}{c^2}\eta = 0$$

for which a solution is (reference 5, pp. 258-260)

$$\theta = C_1 J_o\left(\sqrt{\frac{4B}{c^2}}\, e^{\pm\frac{c}{2}t}\right) + C_2 Y_o\left(\sqrt{\frac{4B}{c^2}}\, e^{\pm ct}\right) \tag{11}$$

where the plus sign corresponds to descent and the minus sign to ascent.

Figure 1 illustrates the motion described by equation (11) for descending flight. The particular case used as an example corresponds to $t = 0$ at an altitude of 400,000 feet. The other assumptions are

$$V = 20,000 \text{ ft/sec}$$

$$\frac{W}{S} = 200 \text{ lb/sq ft}$$

$$\left(\frac{l}{K_g}\right)^2 = \frac{1}{4}$$

$$C_{m_\theta} = -0.1 \times 4 = -0.4$$

$$\frac{V}{l} = 2000 \text{ reference lengths per second}$$

$c = 0.05$ (corresponding to a rate of descent, $V_h = 1360$ feet per second)

It was assumed that at $t = 0$

$$\theta = \theta_o$$

$$\dot{\theta} = 0$$

With these assumed values and initial conditions

$$\theta = 1.215 J_o\left(0.716\ e^{\pm 0.025t}\right) + 0.377 Y_o\left(0.716\ e^{\pm 0.025t}\right)$$

Now

$$J_o(x) \cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$$

and

$$Y_o(x) \cong \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right)$$

Equation (11) can be replaced by

$$\theta = \sqrt{\frac{2}{\pi\left(\sqrt{\frac{4B}{c^2}}\, e^{\pm\frac{c}{2}t}\right)}}\left[C_1 \cos\left(\sqrt{\frac{4B}{c^2}}\, e^{\pm\frac{c}{2}t} - \frac{\pi}{4}\right) + \right.$$

$$\left. C_2 \sin\left(\sqrt{\frac{4B}{c^2}}\, e^{\pm\frac{c}{2}t} - \frac{\pi}{4}\right)\right] \qquad (12)$$

for practical purposes except near $t = 0$ where the Bessel's function must be retained for accuracy.

From equation (12) it is apparent that the maximum amplitude in descending flight is proportional to $e^{-\frac{c}{4}t}$. The variation of energy with time is, therefore

$$E = \frac{I}{2} Be^{ct}\theta_m{}^2 = Ke^{\frac{c}{2}t}$$

indicating a growth of energy with time due to the increase of density in descending flight. Similarly in ascending flight

$$E = Ke^{-\frac{c}{2}t}$$

and the energy in the oscillation decreases with time, even though the amplitude increases in proportion to $e^{\frac{c}{4}t}$.

No useable exact solution was found for equation (5). A practical solution was found, however, as follows:

Equation (12) gives a close approximation to the exact value for θ given by (11) except very near $t = 0$. When this expression for θ is inserted in $\ddot{\theta} + Be^{ct}\theta = 0$ it is found that, since

$$\frac{\theta}{\sqrt{\frac{2}{\pi}}\sqrt[4]{\frac{c^2}{4B}}} = e^{-\frac{c}{4}t}\left[C_1 \cos\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right) + C_2 \sin\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right)\right]$$

Letting

$$\sigma_1 = \left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right)$$

$$\frac{\dot{\theta}}{\sqrt{\frac{2}{\pi}}\sqrt[4]{\frac{c^2}{4B}}} = -\frac{c}{4}\, e^{-\frac{c}{4}t}\left[C_1 \cos \sigma_1 + C_2 \sin \sigma_1\right] + \sqrt{B}\, e^{\frac{c}{4}t}\left[-C_1 \sin \sigma_1 + C_2 \cos \sigma_1\right]$$

and

$$\frac{\ddot{\theta}}{\sqrt{\frac{2}{\pi}}\sqrt[4]{\frac{c^2}{4B}}} = \frac{c^2}{16} e^{-\frac{c}{4}t} C_1\left[\cos \sigma_1 + C_2 \sin \sigma_1\right] -$$

$$\sqrt{\frac{4B}{c^2}}\, \frac{c^2}{8}\, e^{\frac{c}{4}t}\left[-C_1 \sin \sigma_1 + C_2 \cos \sigma_1\right] +$$

$$\frac{c}{4}\sqrt{B}\, e^{\frac{c}{4}t}\left[-C_1 \sin \sigma_1 + C_2 \cos \sigma_1\right] +$$

$$\sqrt{\frac{4B}{c^2}}\sqrt{B}\, \frac{c}{2}\, e^{\frac{3c}{4}t}\left[-C_1 \cos \sigma_1 - C_2 \sin \sigma_1\right]$$

then

$$\left[\left(\frac{c^2}{16} e^{-\frac{c}{4}t} - Be^{\frac{3c}{4}t}\right) + Be^{\frac{3c}{4}t}\right]\left[C_1 \cos \sigma_1 + C_2 \sin \sigma_1\right] +$$

$$\left[-\frac{c}{4}\sqrt{B}\, e^{\frac{c}{4}t} + \frac{c}{4}\sqrt{B}\, e^{\frac{c}{4}t}\right]\left[-C_1 \sin \sigma_1 + C_2 \cos \sigma_1\right] = 0$$

This can be exactly true only if $\frac{c^2}{16} e^{-\frac{c}{4}t} = 0$. Since c is small and $e^{-\frac{c}{4}t}$ decreases rapidly from 1 as t increases, it is apparent why (12) can closely approximate (11), especially at large values of t. The term $\frac{c^2}{16} e^{-\frac{c}{4}t}$ is small relative to $Be^{\frac{3c}{4}t}$ and $\left(\frac{c^2}{16} e^{-\frac{c}{4}t} - Be^{\frac{3c}{4}t}\right)$ asymptotes $-Be^{\frac{3c}{4}t}$ as t becomes large.

If one writes the alternative form

$$\frac{\theta}{\sqrt{\frac{2}{\pi}}} = g^{-\frac{1}{2}}\left[C_1 \cos\left(g - \frac{\pi}{4}\right) + C_2 \sin\left(g - \frac{\pi}{4}\right)\right]$$

then, letting

$$\sigma_2 = \left(g - \frac{\pi}{4}\right)$$

$$\frac{\dot{\theta}}{\sqrt{\frac{2}{\pi}}} = -\frac{1}{2} g^{-\frac{3}{2}} g'\left[C_1 \cos \sigma_2 + C_2 \sin \sigma_2\right] +$$

$$g^{-\frac{1}{2}} g' \left[-C_1 \sin \sigma_2 + C_2 \cos \sigma_2\right]$$

and

$$\frac{\ddot{\theta}}{\sqrt{\frac{2}{\pi}}} = \left[\frac{3}{4}\, g^{-\frac{5}{2}}(g')^2 - \frac{1}{2}\, g^{-\frac{3}{2}} g'' - g^{-\frac{1}{2}}(g')^2\right] \times$$

$$\left[C_1 \cos \sigma_2 + C_2 \sin \sigma_2\right] +$$

$$\left[-g^{-\frac{3}{2}}(g')^2 + g^{-\frac{1}{2}} g''\right] \times$$

$$\left[-C_1 \sin \sigma_2 + C_2 \cos \sigma_2\right]$$

One finds that

$$\left[\frac{3}{4}\, g^{-\frac{5}{2}}(g')^2 - \frac{1}{2}\, g^{-\frac{3}{2}} g''\right] = \frac{c^2}{16}\, e^{-\frac{c}{4}t}$$

which can be neglected, as was previously shown.

Also

$$\left[-g^{-\frac{3}{2}}(g')^2 + g^{-\frac{1}{2}} g''\right] = \left[-\frac{c}{4}\ \sqrt{B}\ e^{\frac{c}{4}t} + \frac{c}{4}\ \sqrt{B}\ e^{\frac{c}{4}t}\right] = 0$$

In order to find an expression for θ to fit equation (5), it seems reasonable that

$$\frac{\theta}{\sqrt{\frac{2}{\pi}}} = f(t)\left[g_1(t)\right]^{-\frac{1}{2}}\left[C_1 \cos\left(g_1 - \frac{\pi}{4}\right) + C_2 \sin\left(g_1 - \frac{\pi}{4}\right)\right]$$

where $f(t)$ is an added damping term and $g_1(t)$ is a new function differing from $g(t)$ only slightly because of the influence of the damping on the period. Under these conditions, it is also reasonable to assume that

$$\left[\frac{3}{4}\, g_1^{-\frac{5}{2}}(g_1')^2 - \frac{1}{2}\, g_1^{\frac{3}{2}} g_1''\right]$$

and

$$\left[-g_1^{-\frac{3}{2}}(g_1')^2 + g_1^{-\frac{1}{2}} g_1''\right]$$

are small and can be neglected. It will be shown later that this assumption is justified. Differentiating $\frac{\theta}{\sqrt{\frac{2}{\pi}}}$ and letting

$$\sigma_3 = \left(g_1 - \frac{\pi}{4}\right)$$

gives

$$\frac{\dot{\theta}}{\sqrt{\frac{2}{\pi}}} = f' g_1^{-\frac{1}{2}}\left[C_1 \cos \sigma_3 + C_2 \sin \sigma_3\right] +$$

$$f\left\{-\frac{1}{2} g_1^{-\frac{3}{2}} g_1' \left[C_1 \cos \sigma_3 + C_2 \sin \sigma_3\right] + \right.$$

$$\left. g_1^{-\frac{1}{2}} g_1' \left[-C_1 \sin \sigma_3 + C_2 \cos \sigma_3\right]\right\}$$

Differentiating again

$$\frac{\ddot{\theta}}{\sqrt{\frac{2}{\pi}}} = f'' g_1^{-\frac{1}{2}}\left[C_1 \cos \sigma_3 + C_2 \sin \sigma_3\right] +$$

$$2f'\left\{-\frac{1}{2} g_1^{-\frac{3}{2}} g_1' \left[C_1 \cos \sigma_3 + C_2 \sin \sigma_3\right] + \right.$$

$$\left. g_1^{-\frac{1}{2}} g_1' \left[-C_1 \sin \sigma_3 + C_2 \cos \sigma_3\right]\right\} +$$

$$f\left\{\left[\frac{3}{4} g_1^{-\frac{5}{2}} (g_1')^2 - \frac{1}{2} g_1^{-\frac{3}{2}} g_1'' - g_1^{-\frac{1}{2}} (g_1')^2\right] \times \right.$$

$$\left[C_1 \cos \sigma_3 + C_2 \sin \sigma_3\right] +$$

$$\left[-\frac{1}{2} g_1^{-\frac{3}{2}} (g_1')^2 - \frac{1}{2} g_1^{-\frac{3}{2}} (g_1')^2 + g^{-\frac{1}{2}} g_1''\right] \times$$

$$\left[-C_1 \sin \sigma_3 + C_2 \cos \sigma_3\right]$$

Neglecting the terms previously shown to be small, and substituting in the original complete differential equation

$$\left[f''g_1^{-\frac{1}{2}} - f'g_1^{-\frac{3}{2}}g_1' - fg_1^{-\frac{1}{2}}(g_1')^2 + Ae^{ct}\left(f'g_1^{-\frac{1}{2}} - \frac{1}{2}g_1^{-\frac{3}{2}}g_1'\right) + Be^{ct}fg_1^{-\frac{1}{2}}\right]\left[C_1 \cos\sigma_3 + C_2 \sin\sigma_3\right] +$$

$$\left[2f'g_1^{-\frac{1}{2}}g_1' + Ae^{ct}fg_1^{-\frac{1}{2}}g_1'\right]\left[-C_1 \sin\sigma_3 + C_2 \cos\sigma_3\right] = 0$$

In order that the equation be satisfied, it is necessary that the coefficients of

$$\left[C_1 \cos\sigma_3 + C_2 \sin\sigma_3\right]$$

and

$$\left[-C_1 \sin\sigma_3 + C_2 \cos\sigma_3\right]$$

each be equal to zero. Therefore

$$f'' - f'g_1g_1' - f(g_1')^2 + Ae^{ct}f' - Ae^{ct}\frac{1}{2}fg_1g_1' + Be^{ct}f = 0$$

and

$$2f' + Ae^{ct}f = 0$$

From the latter relation

$$\frac{f'}{f} = -\frac{A}{2} e^{ct}$$

$$\log f = -\frac{A}{2c} e^{ct} + \text{Const}$$

or

$$f = \text{Const}\; e^{-\frac{A}{2c}e^{ct}}$$

or, allowing the constant to be absorbed in C_1 and C_2

$$f = e^{-\frac{A}{2c}e^{ct}}$$

$$f' = -\frac{A}{2} e^{ct} e^{-\frac{A}{2c}e^{ct}}$$

$$f'' = -\frac{cA}{2} e^{ct} e^{-\frac{A}{2c}e^{ct}} + \left(\frac{A}{2}\right)^2 e^{ct} e^{-\frac{A}{2c}e^{ct}}$$

One can then write

$$-\frac{cA}{2} e^{ct} e^{-\frac{A}{2c}e^{ct}} + \left(\frac{A}{2}\right)^2 e^{ct} e^{-\frac{A}{2c}e^{ct}} + \frac{A}{2} e^{ct} e^{-\frac{A}{2c}e^{ct}} g_1 g_1' -$$

$$e^{-\frac{A}{2c}e^{ct}} (g_1')^2 + Ae^{ct}\left(-\frac{A}{2} e^{ct} e^{-\frac{A}{2c}e^{ct}}\right) -$$

$$Ae^{ct} \frac{1}{2} e^{-\frac{A}{2c}e^{ct}} g_1 g_1' + Be^{ct} e^{-\frac{A}{2c}e^{ct}} = 0$$

or

$$\left[\left(\frac{A}{2}\right)^2 - \frac{cA}{2}\right]e^{ct} - (g_1')^2 - \frac{(Ae^{ct})^2}{2} + Be^{ct} = 0$$

Now in general $\left[\left(\frac{A}{2}\right)^2 - \frac{cA}{2}\right]$ is negligible relative to B and one can write

$$(g_1')^2 = \left[B - \frac{1}{2}\left(Ae^{\frac{c}{2}t}\right)^2\right]e^{ct}$$

or

$$g_1 = \int \left[B - \frac{1}{2}\left(Ae^{\frac{c}{2}t}\right)^2\right]^{1/2} e^{\frac{c}{2}t}\, dt$$

Letting

$$\sigma_4 = \left[\frac{4B}{c^2} - 2\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^2\right]^{1/2}$$

$$g_1 = \frac{c}{2}\int \sigma_4 e^{\frac{c}{2}t}\, dt$$

Integrating by parts gives

$$g_1 = \left\{\sigma_4 + \frac{2}{3}\,\frac{\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^2}{\sigma_4} \quad \frac{4}{15}\,\frac{\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^4}{\sigma_4^{\,3}} \quad \frac{8}{35}\,\frac{\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^6}{\sigma_4^{\,5}} - \right.$$

$$\left. \frac{16}{63}\,\frac{\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^8}{\sigma_4^{\,7}} + \ldots\ldots\ldots \right\} e^{\frac{c}{2}t}$$

In the sample problem, the value of t at which the aircraft has reached the ground is $t = \frac{400,000}{1360} =$ 294 seconds. Also

$$B = 3.2 \times 10^{-4}$$

$$A = 2 \times 10^{-6} \qquad \left(C_{m_{\dot\theta}} = -10\right)$$

$$C = 0.05$$

$$\frac{A}{c} = 4 \times 10^{-5}$$

$$\frac{4B}{c^2} = 0.512$$

$$\left(\frac{A}{c}\, e^{\frac{c}{2}t}\right)^2 = 0.0067$$

$$\sigma_4 = \left[0.512 - 0.0134\right]^{\frac{1}{2}}$$

Inserting the above values in the expression for g_1

$$g_1 = \left\{ \left[0.512 - 0.0134\right]^{\frac{1}{2}} + \frac{\frac{2}{3}(0.0134)}{\left[0.512 - 0.0134\right]^{\frac{1}{2}}} - \frac{\frac{4}{15}(0.0134)^2}{\left[0.512 - 0.0134\right]^{\frac{3}{2}}} + \dots\dots\dots \right\} e^{\frac{c}{2}t}$$

It is apparent that

$$g_1 = \sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} = g$$

can be used without important loss of accuracy. This confirms the assumption that

$$\left[\frac{3}{4}\, g_1^{-\frac{5}{2}}\left(g_1'\right)^2 - \frac{1}{2}\, g_1^{-\frac{3}{2}} g_1''\right]$$

and

$$\left[-g_1^{-\frac{3}{2}}\left(g_1'\right)^2 + g_1^{-\frac{1}{2}} g_1''\right]$$

can be neglected since it was shown earlier that when $g_1 = g$ these quantities are negligible. For most cases of practical interest

$$\theta = e^{-\frac{A}{2c}e^{ct}} \sqrt{\frac{2}{\pi\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t}}} \left[C_1 \cos\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right) - C_2 \sin\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right)\right] \qquad (13)$$

will sufficiently accurately represent the motion. It will be seen that this differs from equation (11) only by the addition of the damping term

$$e^{-\frac{A}{2c}e^{ct}}$$

Although equation (13) is most generally applicable and represents the normal situation of low aerodynamic damping and a moderate aerodynamic spring constant, it is of interest to consider a special case representing a high ratio of damping to spring constant as might exist for an aircraft with very low "static" stability. The special case represented by

$$A = \frac{2B}{c}$$

has the closed solution (reference 6, pp. 166-167)

$$\theta = e^{-\frac{A}{2c}e^{ct}}\left[C_3 I_o\left(\frac{A}{2c}\, e^{ct}\right) + C_4 K_o\left(\frac{A}{2c}\, e^{ct}\right)\right] \qquad (14)$$

For positive values of B the term in K_o is obviously a rapid convergence. The function I_o diverges rapidly but it can readily be shown that the function $\theta = e^{-\frac{A}{2c}e^{ct}} C_3 I_o\left(\frac{A}{2c}\, e^{ct}\right)$ also converges, although slowly.

CHAPTER IV

DISCUSSION

The general disturbed motion of an hypothetical single-degree-of-freedom aircraft entering the earth's atmosphere at constant speed can be approximated by equation (13)

$$\theta = e^{-\frac{A}{2c}e^{ct} - \frac{c}{4}t} \sqrt{\frac{2}{\pi\sqrt{\frac{4B}{c^2}}}} \left[C_1 \cos\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right) + \right.$$

$$\left. C_2 \sin\left(\sqrt{\frac{4B}{c^2}}\, e^{\frac{c}{2}t} - \frac{\pi}{4}\right)\right]$$

where A and B depend on the air density at $t = 0$ while C_1 and C_2 are constants of integration fixed by the characteristics of the motion at $t = 0$.

The motion described by equation (13) differs from Figure 1 only by a more rapid decrease of amplitude with time. It differs from the oscillatory motion described by equation (7a), corresponding to a single-degree-of-freedom aircraft at constant altitude, in that both the aerodynamic damping and the frequency in descending motion increase exponentially with time. Also the motion decays exponentially with time even when the aerodynamic damping is zero. This latter effect arises from the failure of the increase in density with time to feed energy into the oscillatory motion fast enough to support a constant amplitude. The energy in the oscillation can be expressed as

$$E = \frac{I}{2}\left[Be^{ct}(\theta)^2 + (\dot{\theta})^2\right]$$

Since $\dot{\theta} = 0$ at times of maximum θ the energy would be required to increase as e^{ct} to maintain a constant amplitude. Even when $A = 0$ the amplitude decreases as $e^{-\frac{c}{4}t}$ hence the energy increases only as $e^{\frac{c}{2}t}$.

Because of the exponential variation of the aerodynamic damping with time, the total damping at high altitude is almost entirely due to the rate of increase of density. On the other hand, the aerodynamic damping may become very powerful at low altitude, as was the case for the illustrative example.

Equation (13) applies equally well for ascending flight when c is replaced by $(-c)$. It can be written

$$\theta = e^{\left(\frac{A}{2c}e^{-ct} + \frac{c}{4}t\right)} \sqrt{\frac{2}{\pi\sqrt{\frac{4B}{c^2}}}} \left[C_1 \cos\left(\sqrt{\frac{4B}{c^2}}\, e^{-\frac{c}{2}t} - \frac{\pi}{4}\right) + C_2 \sin\left(\sqrt{\frac{4B}{c^2}}\, e^{-\frac{c}{2}t} - \frac{\pi}{4}\right)\right]$$

It will be seen that when A is small, or t is large, the motion corresponds to an unstable oscillation of increasing period. Hence an hypothetical single-degree-of-freedom aircraft leaving the atmosphere at constant speed will tend to perform increasing oscillations when it reaches high altitudes and will obviously take on a tumbling motion, if not otherwise stabilized, as it leaves the atmosphere.

The special case represented by equation (14) indicates the effect of unusually high aerodynamic damping or an unusually low aerodynamic spring constant or a very large rate of vertical descent. The motion under such condition degenerates into a pair of convergences: one very rapid, corresponding to the tendency for $\dot{\theta}$ to be damped quickly to a low value after a disturbance; the other very slow, corresponding to a slow return of θ toward zero after a disturbance.

The preceding analysis and discussion apply only to a single-degree-of-freedom system. As is well known (for example, see the discussion in reference 7) the motion of a multiple-degree-of-freedom system may be undamped although damping of each of the individual degrees of freedom may be positive. This arises from the existence of phase relationships between the motions in the different degrees of freedom which in turn produce situations whereby a motion in translation, for example, feeds energy into a motion of rotation through the spring constant of the rotation. It is probable that the damping of a multiple-degree-of-freedom system will be affected by the change of density with time similarly to the single-degree-of-freedom system. This will be true if the phase relationships are not adversely affected by the change in density with time. Whether they will or will not be so affected can only be determined by simultaneous solution of the differential equations with variable coefficients describing each of the degrees of freedom. This solution has not been attempted during this investigation but obviously should be carried out.

CHAPTER V

SUMMARY

The differential equation for the motion of an hypothetical single-degree-of-freedom aircraft flying at high speed on an inclined path has been solved: exactly for the case of zero aerodynamic damping; approximately for the case when aerodynamic damping is present.

The solutions indicate that in descending flight a disturbance results in an oscillation, the frequency and damping of which increase exponentially with time. In ascending flight following a disturbance, the period increases exponentially with time. The amplitude may at first decrease with time but will eventually increase with time and the vehicle will leave the atmosphere with a tumbling motion.

The behaviour of the single-degree-of-freedom system is probably typical of a multiple-degree-of-freedom system but that this cannot be arbitrarily assumed to be the case has been indicated and should be the subject of further study.

Langley

REFERENCES

APPENDIX I

APPENDIX I

HYPERGEOMETRIC SOLUTION FOR THE DIFFERENTIAL EQUATION $\ddot{\theta} + Ae^{ct}\dot{\theta} + Be^{ct}\theta = 0$

The exact solution of equation (5) is the sum of two related hypergeometric series and was found to give results in agreement with equation (11), the exact solution with $A = 0$, for values of t from 40 to 80 seconds using the values of A, B, and c for the sample problem. In this part of the motion the term $e^{-\frac{A}{2c}e^{ct}}$ is negligible so that the complete solution should agree with the solution neglecting aerodynamic damping.

The hypergeometric series proved to be useless for larger values of t because of their slow convergence and because they involved accurately determining small differences between very large quantities. The hypergeometric solution is presented here merely for the information of the interested reader. The development is as follows: In the equation

$$\ddot{\theta} + Ae^{ct}\dot{\theta} + Be^{ct}\theta = 0 \qquad \text{(A-1)}$$

Let

$$\theta(t) = \eta(\xi)$$

where

$$\xi = e^{ct}$$

This gives

$$\xi\eta'' + \left(1 + \frac{A}{2c}\xi\right)\eta' + \frac{B}{c^2}\eta = 0 \qquad \text{(A-2)}$$

Let

$$\xi_1 = -\frac{A}{c}\xi$$

and

$$\eta_1(\xi_1) = \eta(\xi)$$

$$\eta' = \eta_1' \frac{d\xi_1}{d\xi} = -\frac{A}{c}\eta_1'$$

$$\eta'' = \left(\frac{A}{c}\right)^2 \eta_1''$$

Substituting in (A-2)

$$-\frac{\xi_1}{\frac{A}{c}}\left(\frac{A}{c}\right)^2\eta_1'' + \left(1 - \xi_1\right)\left(-\frac{A}{c}\right)\eta_1' + \frac{B}{c^2}\eta_1 = 0$$

or

$$\xi_1\eta_1'' + \left(1 - \xi_1\right)\eta_1' - \frac{B}{Ac}\eta_1 = 0 \qquad \text{(A-3)}$$

This is the standard form of the confluent hypergeometric equation. A solution for this equation is given in reference 8, p. 428, as

$$\theta = C_1 F\left(\frac{B}{Ac}, 1, \xi_1\right) + C_2 \left[F\left(\frac{B}{Ac}, 1, \xi_1\right) \log \xi_1 + F*\left(\frac{B}{Ac}, 1, \xi_1\right)\right] \qquad \text{(A-4)}$$

where

$$F\left(\frac{B}{Ac}, 1, \xi_1\right) = 1 + \sum_{k=1}^{\alpha} \frac{\frac{B}{Ac}\left(\frac{B}{Ac} + 1\right) \cdots \left(\frac{B}{Ac} + k - 1\right) \xi_1^{\,k}}{\left(k!\right)^2}$$

and

$$F*\left(\frac{B}{Ac}, 1, \xi_1\right) = \frac{\frac{B}{Ac}\,\xi_1}{\left(1!\right)^2}\left(\frac{1}{\frac{B}{Ac}} - \frac{1}{1} - 1\right) + \frac{\frac{B}{Ac}\left(\frac{B}{Ac} - 1\right)\xi_1^{\,2}}{\left(2!\right)^2}\left(\frac{1}{\frac{B}{Ac}} - \frac{1}{\frac{B}{Ac} + 1} - \frac{1}{1} - \frac{1}{1 + 1} - 1 - \frac{1}{2}\right) + \cdots$$

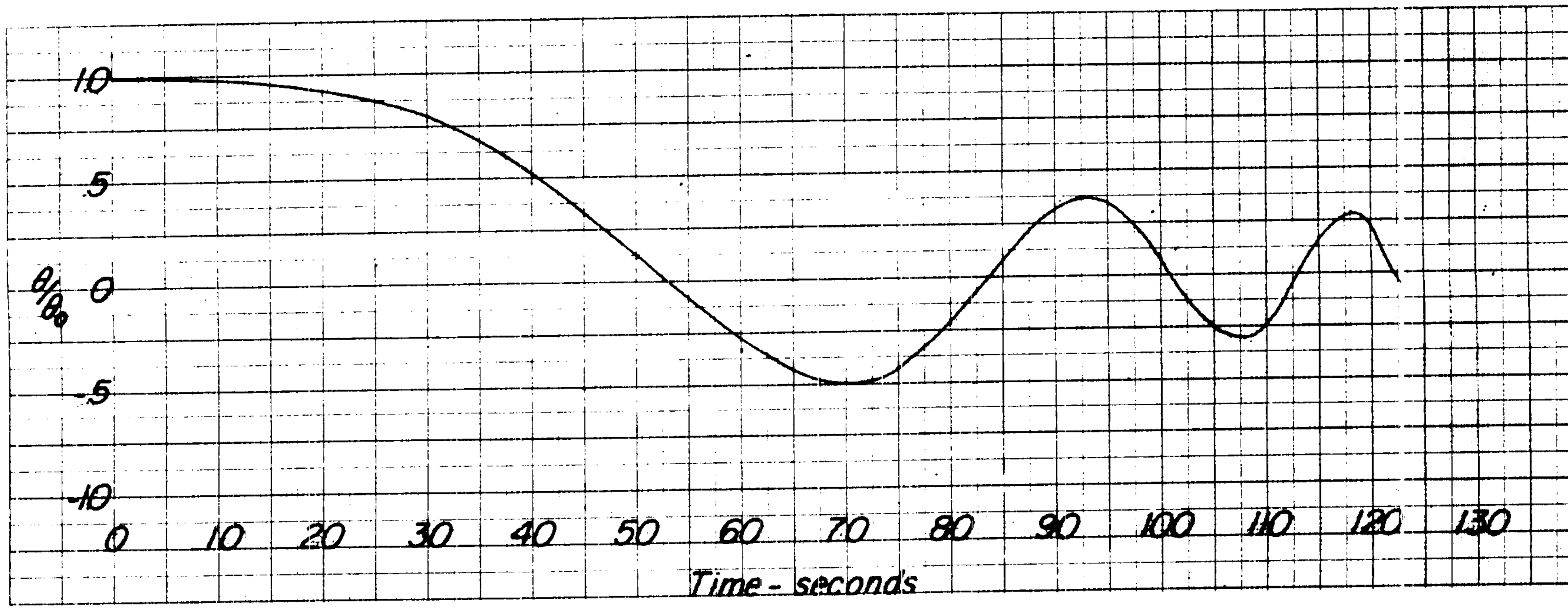

FIGURE 1

VARIATION OF ANGULAR DISPLACEMENT WITH TIME FOR A SINGLE-DEGREE-OF-FREEDOM SYSTEM WITH ZERO AERODYNAMIC DAMPING IN AN ATMOSPHERE THE DENSITY OF WHICH IS INCREASING EXPONENTIALLY WITH TIME